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THE REGENERATIVE HEAT EXCHANGER COMPUTER REPRESENTATION

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Abstract—The adequate representation of the thermal behaviour of the regenerative heat exchanger has attracted the attention of mathematicians for many years. Nearly all efforts have been directed towards the two-dimensional problem, that is it has been required to calculate the temperatures within the regenerator as function of distance, in the direction of gas flow, and of time. The effect of thermal conductivity within heat storing material of the regenerator in a direction perpendicular to the direction of gas flow has either been ignored or incorporated within a lumped or overall heat-transfer coefficient. In this paper, the three-dimensional equations are considered and the effect is discussed of the simplifying assumption that the problem can be regarded to be in two dimensions only. The problem of longitudinal thermal conductivity is not considered since it has been shown that its effect in most practical cases is negligible.

NOMENCLATURE

A, regenerator heating surface area [ft^2, cm^2];
A_2, maximum exit gas temperature, cooling period calculated by 2-D method [°F, °C];
B_2, minimum exit gas temperature, cooling period, calculated by 2-D method [°F, °C];
A_3, maximum exit gas temperature, cooling period, calculated by 3-D method [°F, °C];
B_3, minimum exit gas temperature, cooling period, calculated by 3-D method [°F, °C];
C, specific heat of heat storing matrix [Btu/lb °F, cal/g °C];
d, semi-thickness of wall of heat storing matrix [ft, cm];
h, surface heat-transfer coefficient [Btu/ft^2 h °F, cal/cm^2 s °C];
$\bar{h}$, overall heat-transfer coefficient [Btu/ft^2 h °F, cal/cm s °C];
K/K_0, Hausen ratio defining extent of the non-linear temperature changes in the regenerator;
L, length of regenerator [ft, cm];
M, mass of heat storing matrix [lb, g];
m, mass of residual gas in the regenerator [lb, g];
N, Biot modulus, hd/λ;
P, length of period of operation [h, s];
p, stability parameter, $\alpha\Delta\theta/\Delta x^2 = \Delta w/\Delta z^2$ in the numerical solution of the differential equations;
S, specific heat of gas [Btu/lb °F, cal/g °C];
T, temperature of heat storing matrix [°F, °C];
T_m, mean (in x direction) temperature of heat storing matrix [°F, °C];
t, gas temperature [°F, °C];
thi, entrance gas temperature in the heating period [°F, °C];
tci, entrance gas temperature in the cooling period [°F, °C];
thm, time mean exit gas temperature in the heating period [°F, °C];

tcm, time mean exit gas temperature in the cooling period [°F, °C];
W, flow rate of gas [lb/h, g/s];
w, dimensionless time;
x, distance from surface of heat storing mass in a direction perpendicular to gas flow [ft, cm];
y, distance from regenerator entrance in a direction parallel to gas flow [ft, cm];
z, dimensionless distance from surface of heat storing matrix in a direction perpendicular to gas flow.

Greek symbols
α, thermal diffusivity of heat storing matrix [ft^2/h, cm^2/s];
η, dimensionless time (two dimensional problem);
η_{REG}, thermal ratio;
θ, time [h, s];
λ, thermal conductivity of heat storing matrix [Btu/ft h °F, cal/cm s °C];
Λ, reduced length (three dimensional problem);
$\overline{\Lambda}$, reduced length (two dimensional problem);
μ, dimensionless length in direction of gas flow (two dimensional problem);
ξ, dimensionless length in direction of gas glow (three dimensional problem);
$\overline{\Pi}$, reduced period (two dimensional problem);
ρ, density of heat storing matrix [lb/ft^3, g/cm^3];
ψ, dimensionless ratio $(A_2 - B_2)/(A_3 - B_3)$;
ϕ, correction applied to account for the inversion of the parabolic solid temperature profile at the regenerator reversals;
Ω, reduced time (three dimensional problem);

Subscripts
j, refers to position in y direction;
r, refers to position in x direction;
s, refers to position in time θ;
m, refers to semi-thickness of matrix wall;
0, refers to surface of matrix wall.

Superscripts
$'$, refers to heating period;
$''$, refers to cooling period.

INTRODUCTION

The regenerator consists of a heat storing matrix, often called "chequerwork". Hot gas passes through the channels of the chequerwork and gives up part of its heat to the matrix material. The hot gas is then switched off and cold gas is passed through the same channels and the heat is regenerated from the chequerwork to this gas. In due course, another reversal occurs when the cold gas is shut off and hot gas is passed through the chequerwork channels again.

There are thus three processes of heat transfer which take place in a thermal regenerator.

(i) Heat is transferred within the chequerwork material. This is represented by the diffusion equation:

$$\frac{\partial T}{\partial \theta} = \alpha\left\{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial Z^2}\right\}. \qquad (1)$$

(Here Z is used a direction perpendicular to x and y.)

The thermal conductivity in matrix solid in a direction parallel to that of the gas flow can be shown to be negligible (see Appendix) particularly in the case of chequerwork made of ceramic material, and this is used in most regenerators for industrial application.

(ii) Heat is transferred across the surface of the chequerwork;

(iii) Heat is gained/lost by the gas passing through the regenerator and the gas currently resident in the channels of the chequerwork. The equation is:

$$hA(T_0 - t) = WSL\frac{\partial t}{\partial y} + mS\frac{\partial t}{\partial \theta}. \qquad (2)$$

Equations (1) and (2) express the solid temperature, T, as function of (x, y, z, θ). One simplifying assumption is included, therefore, to reduce the dimensions of the problem. A cross section of part of the regenerator can be represented by Fig. 1. The chequerwork is considered to be a plain wall of specified semi-thickness d, and it is assumed that for any complex chequer shape, the thickness of the "equivalent" plain wall can be calculated. A general picture of a regenerator is to be found in Fig. 2. The problem is thus reduced to the consideration of the solid as a semi-homogenous slab, in which in the direction parallel to gas flow, the conductivity is zero and in the perpendicular direction is finite.

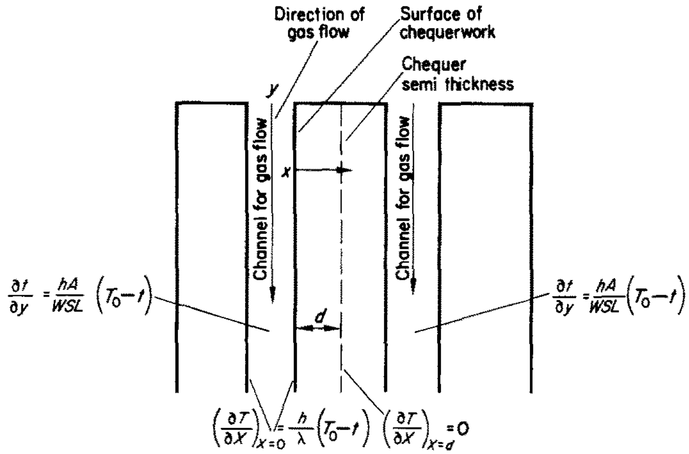

FIG. 1. Illustration of chequerwork cross section and the descriptive differential equations.

Hausen [1] developed methods for dealing with slabs, circular cylinders and spheres, while more recently Razelos and Lazaridis [2] have dealt with the case of the hollow cylinder. Butterfield *et al.* [3] have treated the cases of the hollow cylinder, slab and hollow square section. In these three papers, only a single level in the middle of the regenerator is effectively considered whereas here, simulation of the whole regenerator is attempted.

The differential equation (1) becomes:

$$\frac{\partial T}{\partial \theta} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3}$$

At the surface of the chequerwork, $x = 0$ and $x = 2d$. At the wall semi-thickness, $x = d$. The boundary conditions are specified as:

$$\left.\frac{\partial T}{\partial x}\right|_{x=d} = 0 \tag{4}$$

$$\left.\frac{\partial T}{\partial x}\right|_{\substack{x=0\\x=2d}} = \frac{h}{\lambda}(T_0 - t) \tag{5}$$

At the entrance to the regenerator, $y = 0$, and

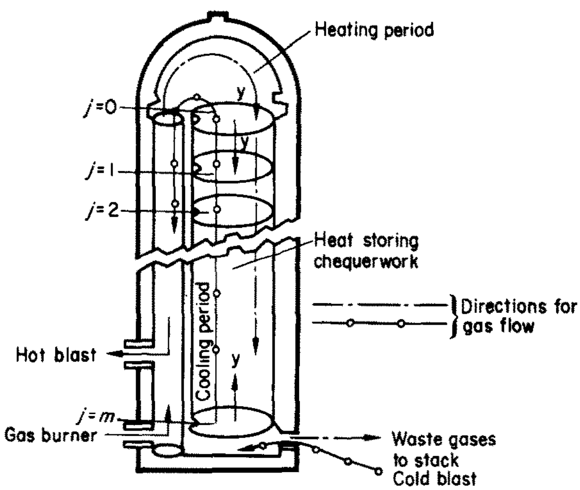

FIG. 2. Schematic drawing of a Cowper stove showing levels in chequerwork for which gas and solid temperatures are calculated.

the gas temperature is considered as not to vary with time.

A regenerator is sometimes called a contra-flow regenerative heat exchanger. The adjective "contra-flow" is applied because the hot gas passes through the regenerator channels in the opposite direction to that of the cold gas. Since the direction y is always measured from the gas entrance of the heat storing mass, the contra-flow operation is incorporated by the equation:

$$T'(x, L - y, 0) = T(x, y, P)$$

where $T'(x, L - y, 0)$ is the chequerwork temperature at the beginning of a period immediately after a reversal and $T(x, y, P)$ is the corresponding temperature immediately before the reversal.

After a large number of successive cycles of identical operation, the regenerator achieves a dynamic equilibrium such that the gas and solid temperature distribution throughout the regenerator is identical at the same point in time in successive cycles. It is for this dynamic or cyclic equilibrium that the solution to the differential equations is often required.

DIMENSIONLESS PARAMETERS

The following replacements are possible in equations (2) and (3).

$$\xi = \frac{hA}{WSL}y; \quad z = \frac{x}{d}; \quad w = \frac{\alpha}{d^2}\left\{\theta - \frac{m}{WL}y\right\}$$

the equations then become:

$$\frac{\partial T}{\partial w} = \frac{\partial^2 T}{\partial z^2} \tag{6}$$

$$\frac{\partial t}{\partial \xi} = T_0 - t. \tag{7}$$

For each period of the cycle, that is the heating period and the cooling period, it is possible to define the descriptive dimensionless parameters, "reduced time" Ω and "reduced length" Λ.

$$\Omega = \frac{\alpha}{d^2}\{P - m/W\}$$

$$\Lambda = \frac{hA}{WS}$$

Ω and Λ correspond to the values of w and ξ when $\theta = P$ and $y = L$.

The boundary conditions represented previously by equations (4) and (5) now become:

$$\left.\frac{\partial T}{\partial z}\right|_{z=1} = 0 \tag{8}$$

$$\frac{\partial T}{\partial z} = \Bigg|_{\substack{z=0\\z=2}} = N(T_0 - t) \tag{9}$$

where $N = hd/\lambda$, the Biot modulus.

The regenerator and its operation can thus be described by six dimensionless parameters and the gas inlet temperatures for the two periods of operation. These are set out in Table 1(a).

Table 1(a)

Heating period	Cooling period	
Ω'	Ω''	Reduced time
Λ'	Λ''	Reduced length
N'	N''	Biot modulus
thi	*tci*	Entrance gas temperature

The single prime is used to refer to the heating period and the double prime to the cooling period.

Methods of solution of the differential equations

There are two basically different approaches to the solution of the equations. The first method is of the closed type where the equations are solved directly for cyclic equilibrium. The unpublished method of Collins and Daws [4] is an example of the closed methods, although Hausen, Iliffe [7] and Nahavandi and Weinstein [5] proposed similar approaches to the two-dimensional regenerator equations. The other procedure, including the ones described here, are of the simulation type where the mathematical model is cycled to equilibrium. Several methods of this type have been described for the equations in two dimensions, including that of Hausen [6] and Willmot [8]. An analogue computer solution to the unsimplified equations used here is described by Ridgion, *et al.* [9]. The calculating of numerical solutions to the differential equations in the three dimensions x, y, and θ, by numerical step by step procedures was made possible by the advent of computers as fast as the University of Manchester ATLAS computer. Without a machine of this speed, the solution of the equations for just one set of operating parameters would have been impracticable.

By solving the differential equations (6) and (7) in finite difference form, the thermal regenerator is simulated and can be cycled to dynamic equilibrium. The numerical solution of equation (6) involves the problem of stability unless special precautions are taken.

We first define p to be $\Delta w/(\Delta z)^2$ where Δw is the time step length and Δz is the distance step length into the chequerwall. If the explicit difference form of equation (6) is employed, that is

$$T_{s+1} = T_s + \Delta w \left.\frac{\partial T}{\partial w}\right|_s$$

or
$$T_{r,s+1} = T_{r,s} + p(T_{r+1,s} - 2T_{r,s} + T_{r-1,s}) \qquad (10)$$

then unless $p \leqslant \frac{1}{2}$, the problem becomes unstable.

In 1947, Crank and Nicolson [10] proposed an application of the trapezoidal formula for the integration of the diffusion equation, namely:

$$T_{s+1} = T_s + \frac{\Delta w}{2}\left\{\left.\frac{\partial T}{\partial w}\right|_{s+1} + \left.\frac{\partial T}{\partial w}\right|_s\right\}$$

or
$$T_{r,s+1} = T_{r,s} + \frac{p}{2}\{T_{r+1,s+1} - 2T_{r,s+1} + T_{r-1,s+1} + T_{r+1,s} - 2T_{r,s} + T_{r-1,s}\} \qquad (11)$$

This yields an implicit scheme in which in order to calculate the values of $T_{r,s+1}$ for $r = 0$ (surface), 1, 2, ... m (brick semi-thickness), it is necessary to solve the resulting algebraic equations, which are essentially of the form

$$ET_{s+1} = FT_s$$

where E and F are tridiagonal matrices.

In order to find $T_{s+1} = E^{-1}FT_s$ involves more labour than the calculation of T_{s+1} by the explicit method. This is more than compensated for by the fact that p can take all values without loss of stability, although as p increases, there is an increase in truncation error.

Mitchell and Pearce [11] pointed out that the Crank–Nicolson method is not the optimum implicit scheme since it does not have the minimum truncation error possible. They specified that the optimum difference representation of equation (6) is:

$$(5 + 6p)T_{r,s+1} + (\tfrac{1}{2} - 3p)(T_{r+1,s+1} + T_{r-1,s+1}) = (5 - 6p)T_{r,s} + (\tfrac{1}{2} + 3p) \times (T_{r+1,s} + T_{r-1,s}). \qquad (12)$$

In order to compute T_{s+1} from T_s using the Mitchell and Pearce scheme involves no more effort than is required for the Crank–Nicolson method, and is equally stable for all values of p.

This method of solving equation (6) has been incorporated into this simulation of the regenerative heat exchanger. It has been modified to take note of the difference representation of the boundary conditions (8) and (9) and equation (7).

At the brick semi-thickness, $r = m$ and $T_{m+1} = T_{m-1}$ since $\partial T/\partial z_m = 0$. At this point, the difference equation becomes

$$(5 + 6p)T_{m,s+1} + (1 - 6p)T_{m-1,s+1} = (5 - 6p)T_{m,s} + (1 + 6p)T_{m-1,s}. \qquad (13)$$

Equation (9), at the solid surface, is presented in the form

$$T_{-1} = T_1 + 2\Delta z\, N(t - T_0).$$

Substituting for T_{-1} in equation (12) yields:

$$\{5 + 6p + (3p - \tfrac{1}{2})(2N\Delta z)\}T_{0,s+1} + (1 - 6p)T_{1,s+1} + (\tfrac{1}{2} - 3p)(2N\Delta z)t_{s+1} = \{5 - 6p - (\tfrac{1}{2} + 3p)(2N\Delta z)\}T_{0,s} + (1 + 6p)T_{1,s} + (\tfrac{1}{2} + 3p)(2N\Delta z)t_s. \qquad (14)$$

Using the notation:

$$A_1 = 5 + 6p + (3p - \tfrac{1}{2})(2N\Delta z)$$
$$B_1 = (\tfrac{1}{2} - 3p)(2N\Delta z)$$
$$A_0 = 5 - 6p - (\tfrac{1}{2} + 3p)(2N\Delta z)$$
$$B_0 = (\tfrac{1}{2} + 3p)(2N\Delta z).$$

This equation becomes:

$$A_1 T_{0,s+1} + (1 - 6p) T_{1,s+1} + B_1 t_{s+1} = A_0 T_{0,s} + (1 + 6p) T_{1,s} + B_0 t_s. \quad (15)$$

Equation (7) can be represented, using the trapezoidal formula in the following form:

$$t_{j+1} = \frac{1-\beta}{1+\beta} t_j + \frac{\beta}{1+\beta}(T_{0,j+1} + T_{0,j}) \quad (16)$$

at any time $s = 0, 1, 2, \ldots$, where $\beta = \Delta\xi/2$.

The finite difference representations of equation (6) set out above are applicable at all levels down the height of the regenerator at $j = 0$ (top of the chequerwork), 1, 2, ..., n (base of the chequers). However, at the entrance ($j = 0$), the gas temperature does not vary with time, that is t_s = constant. Hence at $j = 0$, equation (15) becomes:

$$A_1 T_{0,s+1} + (1 - 6p) T_{1,s+1} = A_0 T_{0,s} + (1 + 6p) T_{1,s} + 6p(2N\Delta z) t_s. \quad (17)$$

At the other levels of the regenerator, $j = 1, 2, \ldots, n$, the gas temperature at time $s + 1$ is obtained using the implicit form:

$$t_{s+1,j+1} - \frac{\beta}{1+\beta} T_{0,s+1,j+1} = \frac{1-\beta}{1+\beta} t_{s+1,j} + \frac{\beta}{1+\beta} T_{0,s+1,j}. \quad (18)$$

These sets of equations are gathered together into matrix form

$$U[T_{s+1}]_{j=0} = V[T_s, t_s]_{j=0} \quad (19)$$

$$G[T_{s+1,j}, t_{s+1,j}] = H(T_{s,j}, t_{s,j}, T_{0,s+1,j-1}, t_{s+1,j-1}] \quad (20)$$

where U, V, G and H are tridiagonal matrices.

The integration process consists of computing $T_{s+1} = U^{-1}V[T_s, t_s]$ at level $j = 0$ and then

$$[T_{s+1}, t_{s+1}] = G^{-1}H[T_{s,j}, t_{s,j}, T_{0,s+1,j-1}, t_{s+1,j-1}]$$

at levels $j = 1, 2, 3, \ldots$ successively to n. This is repeated for $s = 1, 2, \ldots$ until the end of the current period of operation. The solid temperatures at the beginning of the period are specified arbitrarily at the commencement of the integration processes and thereafter by the solid temperatures computed at the end of the previous period.

At the end of a heating period, the direction of gas flow is reversed and in the cooling period, the gas enters the regenerator at $j = n$ (at the base). The method of integration is exactly the same although the matrices U, V, G and H usually will be different and the integration process will proceed from $j = n, n - 1, n - 2$ to 2, 1, 0.

The tridiagonal elements of the matrices U, V, G and H are calculated and stored, and an arbitrary temperature distribution in the heat storing matrix is chosen. The gas temperatures at the beginning of the heating period are then calculated using equation (5) and the specified inlet gas temperature. Since U and G are tridiagonal matrices the process of Gaussian elimination is simplified. Advantage is taken of the fact that the row multipliers need only be calculated once and these are therefore stored at the beginning of the computation. The computer representation of the thermal regenerator, the physical characteristics and operating conditions of which are specified as input data to the program, is cycled to equilibrium or through a preset number of cycles.

Equilibrium is considered to be achieved if successive final exit gas temperatures, heating period, from cycle to cycle, differ by less than 1 in the fifth significant figure.

The explicit method

The explicit finite difference replacement of the diffusion equation is used, namely:

$$T_{r,s+1} = T_{r,s} + p(T_{r+1,s} - 2T_{r,s} + T_{r-1,s}). \quad (10)$$

At the solid surface, when $s = 0$, the boundary condition represented as:

$$T_{-1} = T_1 + 2\Delta z N(t - T_0)$$

is substituted into equation (10) to yield:

$$T_{0,s+1} = T_{0,s} + 2p[T_{1,s} - T_{0,s} + \Delta z N(t - T_{0,s})]. \quad (21)$$

At the semi-thickness of the chequer wall, $r = m$ and $T_{m+1} = T_{m-1}$. Equation (10) becomes

$$T_{m,s+1} = T_{m,s} + 2p(T_{m,s} - T_{m-1,s}). \quad (22)$$

Again, the difference equation (16) is used to represent equation (7).

At the beginning of a heating period of operation, the gas temperatures are calculated using equation (16) at $j = 1, 2, \ldots, n$. The inlet gas temperature is pre-specified as a boundary condition at $j = 0$.

Next, at levels $j = 0, 1, 2, \ldots, n$ the solid temperatures at positions $r = 0, 1, 2, \ldots, m$ into the chequerwork for time Δw, i.e. for $s = 1$, employing equations (10), (21) and (22). This process is again repeated for $s = 2, 3, \ldots$ until the end of the period.

The whole procedure is repeated for the cooling period except that the reversal (counterflow) condition is incorporated. The gas temperatures are computed at $j = m - 1, m - 2, \ldots, 2, 1, 0$ using a suitably modified version of equation (16). The inlet gas temperature is specified as a boundary condition at $j = m$. The integration of equation (6) then proceeds at $j = m, m - 1, m - 2, \ldots, 2, 1, 0$.

This "heating-cooling" cycle is repeated until cyclic equilibrium is reached or for a specified number of cycles.

Although this method, originally discussed by Nusselt [17] as long ago as 1927, is prohibitively slow due to the large number of steps required if stability of the integration process is to be maintained, it is useful for exploratory studies especially if equation (5) involves a non-linear term to account for radiative effects.

Reduction of the number of dimensions

At any height y in the regenerator, the solid temperature is a function of the distance x into the chequerwall and of time θ. The surface of the wall is represented by $x = 0$ and the semi-thickness by $x = d$. At any such height y, the mean solid temperature $T_m(y, \theta)$ is defined by

$$T_m(y, \theta) = \frac{1}{d}\int_0^d T(x, y, \theta)\,dx \quad (23)$$

and an overall heat-transfer coefficient $\bar{h}$ can then be defined by the equation

$$\bar{h}(T_m - t) = h(T_0 - t) \quad (24)$$

where T_0 is the surface solid temperature.

By integrating equation (3), with respect to x Hausen has demonstrated that

$$\frac{\partial T_m}{\partial \theta} = \frac{\bar{h}}{d\rho c}(t - T_m) \quad (25)$$

$Ad\rho = M$, the mass of the chequerwork, and thus equation (25) becomes

$$\frac{\partial T_m}{\partial \theta} = \frac{\bar{h}A}{MC}(t - T_m) \quad (26)$$

Equation (2) is modified to become:

$$\bar{h}A(T_m - t) = WSL\frac{\partial t}{\partial y} + mS\frac{\partial t}{\partial \theta}. \quad (27)$$

If the replacements μ and η are made in equations (26) and (27) where

$$\mu = \frac{\bar{h}Ay}{WSL}, \qquad \eta = \frac{\bar{h}A}{MC}\left(\theta - \frac{m}{WL}y\right)$$

they become

$$\frac{\partial T_m}{\partial \eta} = (t - T_m) \quad (28)$$

$$\frac{\partial t}{\partial \mu} = (T_m - t). \quad (29)$$

Corresponding to each period of the cycle, Hausen proposed the dimensionless parameters, "reduced length", $\bar{\Lambda}$ and "reduced period", $\bar{\Pi}$,

$$\bar{\Lambda} = \bar{h}A/WSL, \qquad \bar{\Pi} = \frac{\bar{h}A}{MC}\left(p - \frac{m}{W}\right).$$

Thus, using this two dimensional representation of the regenerator, there are four dimensionless parameters and the inlet gas temperatures to describe the system, set out in Table 1(b).

Table 1(b)

Heating period	Cooling period	
$\bar{\Lambda}'$	$\bar{\Lambda}''$	Reduced length
$\bar{\Pi}'$	$\bar{\Pi}''$	Reduced period
thi	*tci*	Entrance gas temperature

Various methods have been proposed for the solution of equations (28) and (29), in particular those of Hausen [6], Iliffe [7], Nahavandi and Weinstein [5] and Willmott [8]. However, the adequacy of equations (28) and (29) to represent to the thermal behaviour of a regenerator depends on the use of the lumped (sometimes called "overall") heat-transfer coefficient $\bar{h}$. In his paper of 1942, Hausen [12] suggested that provided the mean solid temperature T_m varies linearly with time, the coefficient $\bar{h}$ can be represented by

$$\bar{h} = (1/h + d\phi/3\lambda)^{-1}$$

where ϕ is the correction factor to account for the inversion of parabolic temperature profile within the chequerwall at the beginning of each period of regenerator operation.

Hausen proposed:

$$\phi = 1 - \frac{d^2}{15\alpha}\left\{\frac{1}{P'} + \frac{1}{P''}\right\}$$

$$\text{if} \quad \frac{d^2}{\alpha}\left\{\frac{1}{P'} + \frac{1}{P''}\right\} \leqslant 5 \qquad (30)$$

$$\phi = \frac{2{\cdot}142}{\sqrt{[0{\cdot}3 + 2d^2(1/P' + 1/P'')]}} \qquad (31)$$

if $\quad (d^2/\alpha)(1/P' + 1/P'') > 5$

or employing the dimensionless parameters,

$$\phi = 1 - \frac{1}{15}\left(\frac{1}{\Omega'} + \frac{1}{\Omega''}\right) \quad \text{if} \quad \frac{1}{\Omega'} + \frac{1}{\Omega''} \leqslant 5 \quad (32)$$

$$\phi = \frac{2{\cdot}142}{\sqrt{[0{\cdot}3 + 2(1/\Omega' + 1/\Omega'')]}}$$

$$\text{if} \; \frac{1}{\Omega'} + \frac{1}{\Omega''} > 5 \qquad (33)$$

Relation between the dimensionless parameters

Reduced length $\bar{\Lambda}$

$$\frac{1}{\bar{\Lambda}} = \frac{WS}{\bar{h}A} = \frac{WS}{A}\left(\frac{1}{h} + \frac{d\phi}{3\lambda}\right) = \frac{1}{\Lambda}\left(1 + \frac{N\phi}{3}\right)$$

Hence:

$$\bar{\Lambda} \; = \; (1 + N\phi/3)^{-1}.$$

Reduced period $\bar{\Pi}$

$$\frac{1}{\bar{\Pi}} = \frac{MC}{\bar{h}AP} = \frac{MC}{AP}\left(\frac{1}{h} + \frac{d\phi}{3\lambda}\right) = \frac{1}{\Omega}\left(\frac{1}{N} + \frac{\phi}{3}\right).$$

Hence: $\quad \bar{\Pi} = \Omega(1/N + \phi/3)^{-1}$

At a position remote from the current regenerator entrance, the mean solid temperature varies linearly with time, but because the gas entrance temperature is constant, the closer to the entrance the more non-linear the time variation of the solid temperature. The non-varying entrance gas temperature can be said to propagate the non-linear behaviour down the regenerator. An estimate of the extent of these non-linearities was suggested by Hausen [12] in form of a factor K/K_0, where $0 < K/K_0 < 1$. The bigger K/K_0, the smaller the effect of the non-linear behaviour of the solid temperature. When $W'S'P' = W''S''P''$, then

$$\frac{K}{K_0} = \frac{\eta_{REG}}{1 - \eta_{REG}}\left(\frac{1}{\bar{\Lambda}'} + \frac{1}{\bar{\Lambda}''}\right). \qquad (34)$$

η_{REG} is the thermal ratio defined for the heating period as

$$\eta'_{REG} = (thi - thm)/(thi - tci)$$

and for the cooling period as

$$\eta''_{REG} = (tcm - tci)/(thi - tci).$$

It is to be noted that *thm* and *tcm* are the chronological mean gas exit temperatures for the heating and cooling period respectively. When $W'S'P' = W''S''P''$, then it is easily verified that at thermal equilibrium, $\eta'_{REG} = \eta''_{REG}$.

This definition of K/K_0 is not completely adequate since it involves $\bar{\Lambda}$, which depends upon ϕ. It will be recalled that ϕ is computed upon the assumption that T_m varies linearly with time whereas K/K_0 seeks to represent the non-linear behaviour of T_m near to the regenerator entrance. However, low values of K/K_0 for a specified value of $\bar{\Lambda}$ are associated with long cycle times, whereas low values of ϕ are associated with short cycle times. In general, therefore, the severe effects of the inversion of the parabolic temperature profile within the chequerwall at the reversals of regenerator operation are not usually felt at the same time as the severe effects of the propagation from the regenerator entrance in either period, of non-linear behaviour of mean solid temperature, T_m.

It might be noted here, however, that the author has found Hausen's K/K_0 factor useful for purposes other than it was intended. Hausen [13] obtained an overall recuperator-like heat exchange coefficient K_0 directly relating the entrance and time mean exit temperatures of the gases in the two periods of operation. The accuracy of the coefficient K_0 depended upon the linear variation of solid temperature with time and in 1942, Hausen [12] introduced the correction factor K/K_0 to account for the propagation of non-linear behaviour by the constant entrance gas temperatures.

Subsequently, Willmott [8] employed the factor K/K_0 as a measure of the truncation errors in the finite difference representation of equations (28) and (29). In this paper, K/K_0 is used as a parameter specifying the extent of non-linear behaviour of solid temperature down the regenerator from the gas entrance.

There is another source of non-linear behaviour of the gas and mean solid temperatures which accentuates the effect of the constant entrance gas temperatures, namely the non-equality of $W'S'P'$ and $W''S''P''$, or in terms of the dimensionless parameters, of $\bar{\Pi}'/\bar{\Lambda}'$ and $\bar{\Pi}''/\bar{\Lambda}''$. In this paper, consideration is restricted to the case when $W'S'P' = W''S''P''$ and, hence, when $\eta'_{REG} = \eta''_{REG}$

Computer solution of the equations

Willmott's method of 1964 has also been programmed and a comparison made between the exit gas temperatures computed by the implicit method for equations (6) and (7) described in this paper and by the 1964 method for the equations (28) and (29).

Clearly, the solution to equations (28) and (29) for dynamic equilibrium should be very close to the corresponding solution to the equations (6) and (7) in three dimensions, *w*, *z*, and ξ. Further, the time taken to solve equations (28) and (29) is considerably less than that to solve the three dimensional problem. The method of computation in the comparison undertaken, has been to solve the equations (28) and (29) for cyclic equilibrium and to employ the calculated solid temperature profile $T_m(\mu)$ at the end of the cycle at dynamic equilibrium as an initial estimate of the solid temperature distribution for the three dimensional problem.

Assuming $\partial T/\partial\theta$ = constant in equation (3)

$$T(x) = T_0 + 0{\cdot}5Kx(x - 2d) \qquad (35)$$

where $T(x)$ is the value of T at x and T_0 is the surface temperature at $x = 0$ and $x = 2d$. K takes the form:

$$K = \frac{1}{\alpha}\frac{\partial T}{\partial\theta} = \frac{\bar{h}}{d\lambda}(t - T_m) \qquad (36)$$

from equation (25); in equation (35), T_0 can be computed from equation (24).

$$T_0 = t + \bar{h}(T_m - t)/h. \qquad (37)$$

At any height of the regenerator, an initial estimation of the temperature profile can therefore be made. T_m is computed by solving equations (28) and (29) for cyclic equilibrium. K and T_0 are then calculated using equations (36) and (24). The temperature profile within the regenerator chequerwork is calculated using equation (35).

The three dimensional equations are then solved for cyclic equilibrium employing this calculated profile as the initial conditions to the problem.

In practice, this has been found to be well worth while, the number of cycles to equilibrium taken in solution of equations (6) and (7) being cut by a factor of at least 4 in comparison with setting $T(x)$ = constant = T_m [computed by solving equations (28) and (29)].

The adequacy of the two dimensional model

Consideration is limited to the case when $\overline{\Pi} = \overline{\Pi}' = \overline{\Pi}''$ and $\overline{\Lambda} = \overline{\Lambda}' = \overline{\Lambda}''$. In 1948, Johnson [14] published his solutions to equations (28) and (29) employing Hausen's heat pole method [6]. Subsequently, Coppage and London [15] recommended for practical use Johnson's values of thermal ratio tabulated at nearly equal intervals of reduced length $\overline{\Lambda}$ (from $\overline{\Lambda} = 5$ to $\overline{\Lambda} = 40$) and of reduced period $\overline{\Pi}$ (from $\overline{\Pi} = 0$ to $\overline{\Pi} = 10$). The figures of Johnson are compared with the figures computed by Willmott's finite differences method, over the range $1 \leqslant \overline{\Pi} \leqslant 3$, $1 \leqslant \overline{\Pi} \leqslant 10$, and this comparison is set out in Table 2. It will be seen that Johnson's figures of η_{REG} diverge from those calculated by the present author for the lower values of K/K_0. The accuracy of Willmott's method has been previously compared with that of Iliffe in a previous paper [8] and has been found to be satisfactory. Table 3 presents the values of K/K_0 and indicates the extent of the non-linear behaviour of temperatures in the regenerator. The truncation error effects associated with the numerical solution of the equations by Willmott's method are minimized by repeating all computations with the time and distance step length's halved if the calculated value of $|\eta'_{REG} - \eta''_{REG}| > 0{\cdot}0000\,1$ and this procedure is incorporated automatically in the

Table 2. Values of thermal ratio, η_{REG} (for $\overline{\Lambda}' = \overline{\Lambda}''$ and $\overline{\Pi}' = \overline{\Pi}''$)
(a) *Computed using Willmott's* [9] *method*
(b) *Recommended figure of Johnson* [8]

Reduced length $\overline{\Lambda}$		Reduced period $\overline{\Pi}$ 1	2	3
1	(a)	0·3221	0·2930	0·2559
	(b)	—	—	—
2	(a)	0·4912	0·4664	0·4305
	(b)	—	—	—
3	(a)	0·5937	0·5757	0·5477
	(b)	—	—	—
4	(a)	0·6622	0·6490	0·6282
	(b)	—	—	—
5	(a)	0·7109	0·7012	0·6856
	(b)	0·7050	0·6950	0·6800
6	(a)	0·7474	0·7400	0·7280
	(b)	0·7430	0·7340	0·7230
7	(a)	0·7758	0·7699	0·7605
	(b)	0·7730	0·7650	0·7550
8	(a)	0·7984	0·7936	0·7861
	(b)	0·7960	0·7890	0·7810
9	(a)	0·8169	0·8129	0·8068
	(b)	0·8150	0·8090	0·8020
10	(a)	0·8322	0·8289	0·8238
	(b)	0·8310	0·8260	0·8200

Table 3. Value of K/K_0 employing Willmott's [9] *figure for η_{REG}*

Reduced length $\overline{\Lambda}$	Reduced period $\overline{\Pi}$ 1	2	3
1	0·9502	0·8287	0·6880
2	0·9652	0·8742	0·7559
3	0·9743	0·9044	0·8072
4	0·9800	0·9247	0·8448
5	0·9838	0·9386	0·8721
6	0·9865	0·9485	0·8922
7	0·9885	0·9558	0·9072
8	0·9900	0·9613	0·9187
9	0·9912	0·9656	0·9278
10	0·9922	0·9691	0·9350

Table 3(a)

Reduced length (2D)	Reduced period (2D)	Hausen factor (phi)	Reduced length (3D)	Reduced time (3D)	Biot modulus (*N*)
1·0000	1·0000	0·9000	1·2903	1·3333	0·9677
2·0000	1·0000	0·9000	2·5806	1·3333	0·9677
3·0000	1·0000	0·9000	3·8710	1·3333	0·9677
4·0000	1·0000	0·9000	5·1613	1·3333	0·9677
5·0000	1·0000	0·9000	6·4516	1·3333	0·9677
6·0000	1·0000	0·9000	7·7419	1·3333	0·9677
7·0000	1·0000	0·9000	9·0323	1·3333	0·9677
8·0000	1·0000	0·9000	10·3226	1·3333	0·9677
9·0000	1·0000	0·9000	11·6129	1·3333	0·9677
10·0000	1·0000	0·9000	12·9032	1·3333	0·9677
1·0000	2·0000	0·9000	1·8182	1·3333	2·7273
2·0000	2·0000	0·9000	3·6364	1·3333	2·7273
3·0000	2·0000	0·9000	5·4545	1·3333	2·7273
4·0000	2·0000	0·9000	7·2727	1·3333	2·7273
5·0000	2·0000	0·9000	9·0909	1·3333	2·7273
6·0000	2·0000	0·9000	10·9091	1·3333	2·7273
7·0000	2·0000	0·9000	12·7273	1·3333	2·7273
8·0000	2·0000	0·9000	14·5455	1·3333	2·7273
9·0000	2·0000	0·9000	16·3636	1·3333	2·7273
10·0000	2·0000	0·9000	18·1818	1·3333	2·7273
1·0000	3·0000	0·9000	3·0769	1·3333	6·9231
2·0000	3·0000	0·9000	6·1538	1·3333	6·9231
3·0000	3·0000	0·9000	9·2308	1·3333	6·9231
4·0000	3·0000	0·9000	12·3077	1·3333	6·9231
5·0000	3·0000	0·9000	15·3846	1·3333	6·9231
6·0000	3·0000	0·9000	18·4615	1·3333	6·9231
7·0000	3·0000	0·9000	21·5385	1·3333	6·9231
8·0000	3·0000	0·9000	24·6154	1·3333	6·9231
9·0000	3·0000	0·9000	27·6923	1·3333	6·9231
10·0000	3·0000	0·9000	30·7692	1·3333	6·9231

computer programme. For most computations, $\Delta\xi = \Lambda/10$ and $\Delta\eta = \Pi/20$ were used.

A source of inadequacy in the two dimensional model is the representation of the inversion of temperature profile within the chequerwall at the reversals. The smaller the value of the correction factor ϕ, the more severe the effect of these inversions.

The equations in the three dimensional form are solved for the values Λ, Ω and N (equal in both heating and cooling periods) corresponding to successive values of $\phi = 0{\cdot}9, 0{\cdot}8, 0{\cdot}7, 0{\cdot}6$, and $0{\cdot}5$ and the possible values of $\overline{\Pi}$ and $\overline{\Lambda}$ chosen in the range $1 \leqslant \overline{\Pi} \leqslant 3$, $1 \leqslant \overline{\Lambda} \leqslant 10$.

The values of Ω, Λ and N are calculated in the following way: When $\Omega = \Omega' = \Omega''$, the value of ϕ is calculated using

$$\phi = 1 - \frac{2}{15\Omega} \quad \text{if} \quad \Omega \geqslant 0{\cdot}4 \tag{38}$$

Hence

$$\Omega = 2/[15(1-\phi)], \text{ unless } \Omega < 0{\cdot}4$$

in which case

$$\Omega = 4/[0{\cdot}3 - (2{\cdot}142/\phi)^2] \tag{39}$$

$$\overline{\Pi} = \Omega(1/N + \phi/3)^{-1} \tag{40}$$

Hence

$$N = (\Omega/\overline{\Pi} - \phi/3)^{-1} \tag{41}$$

It should be noticed immediately that it is possible to calculate, using this formula, infinite and negative values of N which are meaningless.

This points to an important conclusion, namely that for a given value of ϕ, there is an upper value for $\overline{\Pi}$ beyond which it is impossible to relate the two dimensional problem to any

Table 3(b)

Reduced length	Reduced period	Hausen factor (2D)	Reduced length (2D)	Reduced time (3D)	Biot modulus (*N*)
1·0000	1·0000	0·8000	1·6667	0·6667	2·5000
2·0000	1·0000	0·8000	3·3333	0·6667	2·5000
3·0000	1·0000	0·8000	5·0000	0·6667	2·5000
4·0000	1·0000	0·8000	6·6667	0·6667	2·5000
5·0000	1·0000	0·8000	8·3333	0·6667	2·5000
6·0000	1·0000	0·8000	10·0000	0·6667	2·5000
7·0000	1·0000	0·8000	11·6667	0·6667	2·5000
8·0000	1·0000	0·8000	13·3333	0·6667	2·5000
9·0000	1·0000	0·8000	15·0000	0·6667	2·5000
10·0000	1·0000	0·8000	16·6667	0·6667	2·5000
1·0000	2·0000	0·8000	5·0000	0·6667	15·0000
2·0000	2·0000	0·8000	10·0000	0·6667	15·0000
3·0000	2·0000	0·8000	15·0000	0·6667	15·0000
4·0000	2·0000	0·8000	20·0000	0·6667	15·0000
5·0000	2·0000	0·8000	25·0000	0·6667	15·0000
6·0000	2·0000	0·8000	30·0000	0·6667	15·0000
7·0000	2·0000	0·8000	35·0000	0·6667	15·0000
8·0000	2·0000	0·8000	40·0000	0·6667	15·0000
9·0000	2·0000	0·8000	45·0000	0·6667	15·0000
10·0000	2·0000	0·8000	50·0000	0·6667	15·0000

Table 3(c)

Reduced length (2D)	Reduced period (2D)	Hausen factor (phi)	Reduced length (3D)	Reduced time (3D)	Biot modulus (N)
1·0000	1·0000	0·7000	2·1053	0·4444	4·7368
2·0000	1·0000	0·7000	4·2105	0·4444	4·7368
3·0000	1·0000	0·7000	6·3158	0·4444	4·7368
4·0000	1·0000	0·7000	8·4211	0·4444	4·7368
5·0000	1·0000	0·7000	10·5263	0·4444	4·7368
6·0000	1·0000	0·7000	12·6316	0·4444	4·7368
7·0000	1·0000	0·7000	14·7368	0·4444	4·7368
8·0000	1·0000	0·7000	16·8421	0·4444	4·7368
9·0000	1·0000	0·7000	18·9474	0·4444	4·7368
10·0000	1·0000	0·7000	21·0526	0·4444	4·7368
1·0000	1·0000	0·6000	2·6472	0·3214	8·2361
2·0000	1·0000	0·6000	5·2944	0·3214	8·2361
3·0000	1·0000	0·6000	7·9417	0·3214	8·2361
4·0000	1·0000	0·6000	10·5889	0·3214	8·2361
5·0000	1·0000	0·6000	13·2361	0·3214	8·2361
6·0000	1·0000	0·6000	15·8833	0·3214	8·2361
7·0000	1·0000	0·6000	18·5305	0·3214	8·2361
8·0000	1·0000	0·6000	21·1777	0·3214	8·2361
9·0000	1·0000	0·6000	23·8250	0·3214	8·2361
10·0000	1·0000	0·6000	26·4722	0·3214	8·2361
1·0000	1·0000	0·5000	4·0354	0·2216	18·2125
2·0000	1·0000	0·5000	8·0708	0·2216	18·2125
3·0000	1·0000	0·5000	12·1062	0·2216	18·2125
4·0000	1·0000	0·5000	16·1417	0·2216	18·2125
5·0000	1·0000	0·5000	20·1771	0·2216	18·2125
6·0000	1·0000	0·5000	24·2125	0·2216	18·2125
7·0000	1·0000	0·5000	28·2479	0·2216	18·2125
8·0000	1·0000	0·5000	32·2833	0·2216	18·2125
9·0000	1·0000	0·5000	36·3187	0·2216	18·2125
10·0000	1·0000	0·5000	40·3541	0·2216	18·2125

possible corresponding problem in three dimensions. This means, as has been discussed, that it is unlikely that a regenerator problem suffering from the effects of non-linear behaviour (low K/K_0, large $\overline{\Pi}$) will be beset simultaneously by the effects of the regenerator reversals (small ϕ). It follows that the effects of ϕ and K/K_0 upon the adequacy of the two dimensional model can, to some extent, be studied separately.

Finally, in order to calculate the reduced length Λ, the formula below is used.

$$\Lambda = \overline{\Lambda}(1 + N\phi/3) \qquad (42)$$

The possible values of Ω, Λ and N over which a comparison between the two and three dimensional models is made, are set out in

Table 4. *Values of thermal ratio,* η_{REG} *(for* $\overline{\Lambda}' = \overline{\Lambda}''$ *and* $\overline{\Pi}' = \overline{\Pi}''$*)*
(a) *Computed using Willmott's* [9] *method*
(b) *Computed by* 3-D *method,* $\phi = 0{\cdot}9$, $\Omega = 1{\cdot}333$

Reduced length $\overline{\Lambda}$		Reduced period $\overline{\Pi}$ 1	2	3
1	(a)	0·3221	0·2930	0·2559
	(b)	0·3193	0·2849	0·2437
2	(a)	0·4912	0·4664	0·4305
	(b)	0·4890	0·4590	0·4169
3	(a)	0·5937	0·5757	0·5477
	(b)	0·5924	0·5702	0·5360
4	(a)	0·6622	0·6490	0·6282
	(b)	0·6613	0·6453	0·6194
5	(a)	0·7109	0·7012	0·6856
	(b)	0·7105	0·6988	0·6794
6	(a)	0·7474	0·7400	0·7280
	(b)	0·7473	0·7385	0·7239
7	(a)	0·7758	0·7699	0·7605
	(b)	0·7758	0·7691	0·7580
8	(a)	0·7984	0·7936	0·7861
	(b)	0·7986	0·7933	0·7848
9	(a)	0·8169	0·8129	0·8068
	(b)	0·8172	0·8130	0·8064
10	(a)	0·8322	0·8289	0·8238
	(b)	0·8326	0·8293	0·8241

Table 5. *Values of thermal ratio,* η_{REG} *(for* $\overline{\Lambda}' = \overline{\Lambda}''$ *and* $\overline{\Pi}' = \overline{\Pi}''$*)*
(a) *Computed using Willmott's* [9] *method*
(b) *Computed by* 3-D *method* $\phi = 0{\cdot}8$, $\Omega = 0{\cdot}666$

Reduced length $\overline{\Lambda}$		Reduced period $\overline{\Pi}$ 1	2
1	(a)	0·3221	0·2930
	(b)	0·3127	0·2667
2	(a)	0·4912	0·4664
	(b)	0·4836	0·4404
3	(a)	0·5937	0·5757
	(b)	0·5887	0·5553
4	(a)	0·6622	0·6490
	(b)	0·6589	0·6346
5	(a)	0·7109	0·7012
	(b)	0·7089	0·6914
6	(a)	0·7474	0·7400
	(b)	0·7463	0·7337
7	(a)	0·7758	0·7699
	(b)	0·7752	0·7655
8	(a)	0·7984	0·7936
	(b)	0·7982	0·7909
9	(a)	0·8169	0·8129
	(b)	0·8170	0·8126
10	(a)	0·8322	0·8289
	(b)	0·8326	0·8298

Tables 3(a), (b) and (c) for the ranges $1 \leqslant \overline{\Pi} \leqslant 3$ (no higher integral values of $\overline{\Pi}$ are possible) and $1 \leqslant \overline{\Lambda} \leqslant 10$.

In Tables 4–8 are set out the values of the thermal ratio η_{REG} computed by the three dimensional method, together with the corresponding values computed using the two dimensional method, for $\phi = 0{\cdot}9$, 0·8, 0·7, 0·6 and 0·5. For $\phi = 0{\cdot}5$, the two sets of values for η_{REG} are presented graphically in Fig. 4.

The thermal ratio is a dimensionless form of the chronological mean exit gas temperature. If the computed variations with time of this temperature are compared for the two methods, then a significant difference is noted—a difference due to the inadequacy of the ϕ correction in the two dimensional model. The variations take the form indicated in Fig. 3.

Tables 9–13 set out the values of $(A_2 - B_2)/(A_3 - B_3)$ for all the cases considered and they are presented in the form of a graph for $\overline{\Pi} = 1$, in Fig. 5.

Table 6. *Values of thermal ratio,* η_{REG} (*for* $\overline{\Lambda}' = \overline{\Lambda}''$ *and* $\overline{\Pi}' = \overline{\Pi}''$)
(a) *Computing using Willmott's* [9] *method*
(b) *Computed by* 3-D *method* $\phi = 0{\cdot}7$, $\Omega = 0{\cdot}444$

Reduced length $\overline{\Lambda}$		Reduced period $\overline{\Pi}$: 1
1	(a)	0·3221
	(b)	0·3027
2	(a)	0·4912
	(b)	0·4749
3	(a)	0·5937
	(b)	0·5822
4	(a)	0·6622
	(b)	0·6543
5	(a)	0·7109
	(b)	0·7056
6	(a)	0·7474
	(b)	0·7438
7	(q)	0·7758
	(b)	0·7734
8	(a)	0·7984
	(b)	0·7969
9	(a)	0·8169
	(b)	0·8161
10	(a)	0·8322
	(b)	0·8320

Table 7. *Values of thermal ratio,* η_{REG} (*for* $\overline{\Lambda}' = \overline{\Lambda}''$ *and* $\overline{\Pi}' = \overline{\Pi}''$)
(a) *Computed using Willmott's* [9] *method*
(b) *Computed by* 3-D *method* $\phi = 0{\cdot}6$, $\Omega = 0{\cdot}252$

Reduced length $\overline{\Lambda}$		Reduced period $\overline{\Pi}$: 1
1	(a)	0·3221
	(b)	0·2895
2	(a)	0·4912
	(b)	0·4624
3	(a)	0·5937
	(b)	0·5724
4	(a)	0·6622
	(b)	0·6469
5	(a)	0·7110
	(b)	0·7001
6	(a)	0·7475
	(b)	0·7396
7	(a)	0·7758
	(b)	0·7702
8	(a)	0·7984
	(b)	0·7944
9	(a)	0·8169
	(b)	0·8141
10	(a)	0·8322
	(b)	0·8305

In the 3-D computation, $\Delta z = \frac{1}{8}$, and $\Delta \xi = \Lambda/10$ were used. $\Delta \eta$ was chosen so that the stability factor p was less than 1·5. Reduced values of $\Delta \eta$ and $\Delta \xi$ were used if $|\eta'_{REG} - \eta''_{REG}|$ at equilibrium was greater than 0·0001.

Conclusions

The use of the correction factor ϕ in the bulk heat-transfer coefficient, $\bar{h}$, is obtained by considering a mean solid temperature $T_m(y, \theta)$ and by specifying that at any height in the regenerator, the heat transferred per period is unchanged, that is

$$\bar{h} = \frac{h \int_0^P T(x, y, \theta) - t(y, \theta)\, d\theta}{\int_0^P T_m(y, \theta) - t(y, \theta)\, d\theta}, \qquad x = 0$$

Table 8. Values of thermal ratio, η_{REG} (for $\bar{\Lambda}' = \bar{\Lambda}''$ and $\bar{\Pi}' = \bar{\Pi}''$)
(a) *Computed using Willmott's* [9] *method*
(b) *Computed using* 3-D *method* $\phi = 0{\cdot}5, \Omega = 0{\cdot}174$

Reduced length $\bar{\Lambda}$		Reduced period $\bar{\Pi}$ 1
1	(a)	0·3221
	(b)	0·2725
2	(a)	0·4911
	(b)	0·4465
3	(a)	0·5937
	(b)	0·5606
4	(a)	0·6622
	(b)	0·6390
5	(a)	0·7109
	(b)	0·6951
6	(a)	0·7474
	(b)	0·7368
7	(a)	0·7758
	(b)	0·7688
8	(a)	0·7984
	(b)	0·7941
9	(a)	0·8169
	(b)	0·8147
10	(a)	0·8322
	(b)	0·8316

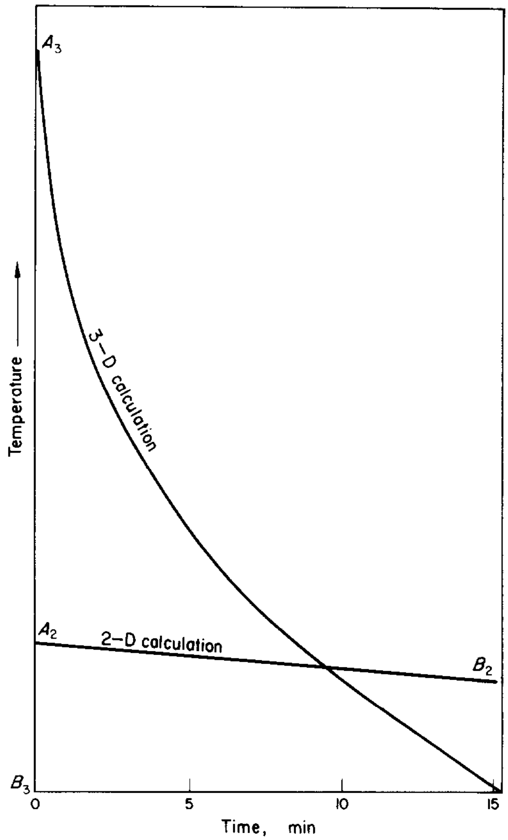

FIG. 3. Comparisons between exit gas temperatures computed using 3-D equations and using 2-D equations.

where

$$T_m(y, \theta) = \frac{1}{d}\int_0^d T(x, y, \theta)\,dx.$$

The time integrals are computed on the assumption that $\partial T_m/\partial\theta$ = constant and the ϕ correction introduced to allow for the inversion of the parabolic profile at the regenerator reversals.

It might be expected, therefore, that for high values of K/K_0, that is when the effect on non-linear changes in T_m with respect to time is smallest, the correlation between the thermal ratios computed by the 3-D method and those by the 2-D method is the best. This correlation deteriorates for decreasing values of K/K_0. This is borne out by our computations and upon examination of Table 3, together with Tables 4–7, it will be observed that the smaller the value of K/K_0, the greater the divergence

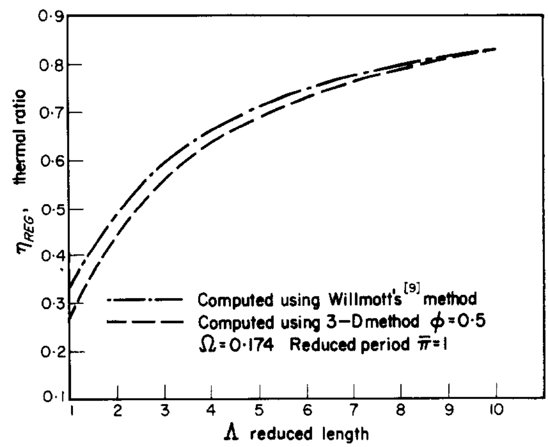

FIG. 4. Comparisons between thermal ratios computed by 2-D and 3-D methods.

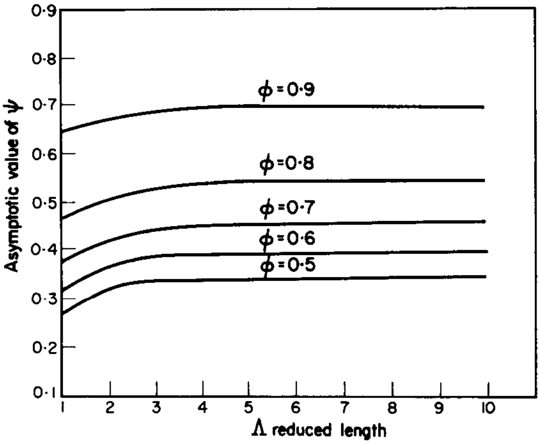

FIG. 5. Relationship between ψ, the ratio of (max. exit temp.—min. exit temp.) 2D to the (max. exit temp.—min. exit temp.) 3D and reduced length . $\bar{\Pi} = 1$.

Table 10. *Values of ratio*:

$$\frac{\text{(Max. exit gas temp} - \text{min. exit gas temp) 2D}}{\text{(Max. exit gas temp} - \text{min. exit gas temp) 3D}}$$

Computed by 3-D *method* $\phi = 0{\cdot}8$, $\Omega = 0{\cdot}666$

Reduced length $\bar{\Lambda}$	Reduced period $\bar{\Pi}$	
	1	2
1	0·4671	0·4502
2	0·5085	0·5667
3	0·5288	0·5987
4	0·5375	0·6073
5	0·5413	0·6105
6	0·5432	0·6133
7	0·5444	0·6162
8	0·5454	0·6190
9	0·5462	0·6235
10	0·5469	0·6273

between the "3-D thermal ratio" and the "2-D thermal ratio", see Fig. 4.

Although for high values of K/K_0 ($>0{\cdot}9$) the 2-D method can be relied upon to provide quite an accurate estimate of time mean exit gas temperature, because $\bar{h}$ involves time means, no accuracy can be placed upon the computed time variation of exit temperature. When $\phi \to 1$, however, $\partial T_m(y, \theta)/\partial\theta$ becomes equal to $\partial T(x, y, \theta)/\partial\theta$ for $0 \leqslant x \leqslant d$ for the most of each period, particularly as $K/K_0 \to 1$.

At the reversals this is not true, and as ϕ decreases, the greater is the effect of these reversals. Tables 9–12 reveal significantly decreasing value of $(A_2 - B_2)/(A_3 - B_3)$ the smaller the value of ϕ. As K/K_0 decreases, $(A_2 - B_2)/(A_3 - B_3)$ also becomes smaller but the effect of K/K_0 is not as significant as the effect of the factor ϕ. As $K/K_0 \to 1$ as reduced length Λ increases, the value of $(A_2 - B_2)/(A_3 - B_3)$ or ψ becomes asymptotic, as shown in Fig. 5.

The three dimensional method of solving the

Table 9. *Values of ratio*:

$$\frac{\text{(Max. exit gas temp} - \text{min exit gas temp) 2D}}{\text{(Max. exit gas temp} - \text{min exit gas temp) 3D}}$$

Computed by 3-D *method* $\phi = 0{\cdot}9$, $\Omega = 1{\cdot}333$

Reduced length $\bar{\Lambda}$	Reduced period Π		
	1	2	3
1	0·6484	0·6179	0·5847
2	0·6726	0·6881	0·7225
3	0·6860	0·7192	0·7687
4	0·6926	0·7313	0·7820
5	0·6956	0·7360	0·7858
6	0·6970	0·7382	0·7877
7	0·6977	0·7397	0·7897
8	0·6983	0·7409	0·7919
9	0·6987	0·7421	0·7944
10	0·6990	0·7432	0·7970

Table 11. *Value of ratio*:

$$\frac{\text{(Max. exit gas temp} - \text{min. exit gas temp) 2D}}{\text{(Max. exit gas temp} - \text{min. exit gas temp) 3D}}$$

Computed by 3-D *method* $\phi = 0{\cdot}7$, $\Omega = 0{\cdot}444$

Reduced length $\bar{\Lambda}$	Reduced period $\bar{\Pi}$
	1
1	0·3726
2	0·4217
3	0·4423
4	0·4501
5	0·4534
6	0·4554
7	0·4569
8	0·4583
9	0·4594
10	0·4606

Table 12. *Value of ratio*:

$$\frac{(\text{Max. exit gas temp} - \text{min. exit gas temp})\,2D}{(\text{Max. exit gas temp} - \text{min. exit gas temp})\,3D}$$

Computed by 3-D *method*, $\phi = 0{\cdot}6$, $\Omega = 0{\cdot}252$

Reduced length $\bar{\Lambda}$	Reduced period $\bar{\Pi}$ 1
1	0·3157
2	0·3676
3	0·3853
4	0·3909
5	0·3934
6	0·3953
7	0·3971
8	0·3989
9	0·4006
10	0·4021

regenerator equations involves the heavy expenditure of computer time and on small computers, the method is impracticable. Further it is possible in practice to develop methods of solving the 2-D equations for non-linear problems such as those involving temperature dependent specific heat and time varying flow rate. Although such developments of the 3-D model are possible in theory, the amount of computer time involved, even on a very fast computer, appears prohibitive.

Table 13. *Value of ratio*:

$$\frac{(\text{Max. exit gas temp} - \text{min. exit gas temp})\,2D}{(\text{Max. exit gas temp} - \text{min. exit gas temp})\,3D}$$

Computed by 3-D *method*, $\phi = 0{\cdot}5$, $\Omega = 0{\cdot}174$

Reduced length $\bar{\Lambda}$	Reduced period $\bar{\Pi}$ 1
1	0·2665
2	0·3206
3	0·3337
4	0·3368
5	0·3387
6	0·3409
7	0·3436
8	0·3463
9	0·3492
10	0·3518

Although consideration here is limited to the plain slab problem, the principles by which the reliability of a 2-D model to represent more complex geometries can be tested are set out. Use of the factors ϕ and K/K_0 will be equally applicable. Hausen [1] has developed formulae for ϕ for other geometric shapes.

In using 2-D methods of solving the regenerator equations, caution should be used in interpreting the computations of $K/K_0 < 0{\cdot}9$ and no reliance should be placed upon the calculated time variations of temperature for values of $\phi < 0{\cdot}9$. For Cowper stoves, typical figures are

$$\alpha = 0{\cdot}02\ \text{ft}^2/\text{h}, \qquad d = 0{\cdot}052\ \text{ft}$$

$$P_1 = 2\ \text{hr}, \qquad P_1 = 1\ \text{hr}, \qquad \phi = 0{\cdot}97$$

indicating caution need only be applied when K/K_0 becomes small (this can vary from 0·7 to 0·95 for Cowper stoves) or if very short cycles are considered which reduces the value of ϕ.

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APPENDIX

An Estimate of the Longitudinal Conductivity in the Heat Storing Mass

The temperature gradient down the regenerator in, for example, the heating period is approximately $(thi - thm)/L$.

The volume of the chequerwork is Ad

Therefore the cross sectional area of chequermass is Ad/L

Therefore the rate of longitudinal heat conduction in the chequerwork is approximately

$$Ad\lambda(thi - thm)/L^2 \text{ Btu/h (cal/s).}$$

Rate of heat input to regenerator by the gas in the heating period is

$$WS\,(thi - thm) \text{ Btu/h (cal/s)}$$

The ratio γ

$$\gamma = \frac{\text{longitudinal heat conduction}}{\text{heat input by gas}} = \frac{Ad\lambda}{L^2 WS}$$

For a Cowper Stove, typical figures are:

$A = 200{,}000 \text{ ft}^2$
$d = 0{\cdot}05 \text{ ft}$
$\lambda = 0{\cdot}7 \text{ Btu/ft h/}^\circ\text{F}$
$L = 80 \text{ ft}$
$W = 200{,}000 \text{ lb/h}$
$S = 0{\cdot}25 \text{ Btu/lb }^\circ\text{F}$

Therefore

$$\gamma = 2{\cdot}2 \times 10^{-5}$$

The figure for thermal conductivity λ refers to a ceramic material.

This implies that longitudinal conduction can be neglected and indeed most authors always assume that a term in the differential equations which accounts for this can be ignored. Tipler [16] mentions a value of 10^{-2} for γ in the case of regenerators used in Gas Turbines and considered that longitudinal conduction could be neglected.

Résumé—La représentation adéquate du comportement thermique de l'échangeur de chaleur par récupération a attiré l'attention des mathématiciens depuis de nombreuses années. Presque tous les efforts ont été dirigés vers les problèmes bidimensionnels, dont on a eu besoin pour calculer les températures dans le récupérateur en fonction de la distance, dans la direction de l'écoulement gazeux, et du temps. L'effet de la conductivité thermique à l'intérieur du matériau d'accumulation de la chaleur du récupérateur dans une direction perpendiculaire à la direction de l'écoulement gazeux a été, soit ignoré, soit incorporé avec un coefficient de transport de chaleur discret ou global. Dans cet article, les équations tridimensionnelles sont prises en considération et l'on discute l'effet de l'hypothèse simplificatrice selon laquelle le problème peut être considéré comme ayant seulement deux dimensions. Le problème de la conductivité thermique longitudinale n'est pas examiné, puisqu'on a montré que son effet dans la plupart des cas pratiques est négligeable.

Zusammenfassung—Die adäquate Beschreibung des thermischen Übertragungsverhaltens von regenerativen Wärmeaustauschern hat schon seit vielen Jahren bei Mathematikern Interesse gefunden. Fast alle Bemühungen haben sich auf das zweidimensionale Problem konzentriert, d.h. man hat sich die Aufgabe gestellt, die Temperaturen im Regenerator als Funktion der Entfernung, gemessen in Strömungsrichtung, sowie der Zeit zu berechnen. Der Einfluss der Wärmeleitfähigkeit in der Speichermasse des Regenerators quer zur Strömungsrichtung ist entweder vernachlässigt, oder in Form eines bestimmten Zusatzwiderstandes dem Wärmeübergangswiderstand zugeschlagen worden. In dieser Arbeit werden die dreidimensionalen Gleichungen zugrunde gelegt und es wird diskutiert, inwieweit sich die Annahme, das Problem könne rein zweidimensional behandelt werden, auswirkt. Das Problem der Läbngwärmeleitung wird nicht weiter untersucht, da gezeigt worden ist, dass dieser Effekt in den meisten praktischen Fällen vernachlässigbar bleibt.

Аннотация—Задача удовлетворительного описания работы регенеративного теплообменника многие годы привлекала внимание математиков. Почти все попытки были направлены на решение двумерных задач, т.е. требовалось рассчитать температуры в регенераторе как функцию расстояния в направлении течения газа и времени. Влияние теплопроводности в материале, аккумулирующем тепло, в направлении перпендикулярном направлению потока газа, либо пренебрегалось, либо учитывалось общим коэффициентом теплообмена. В данной статье рассматриваются трехмерные уравнения и обсуждается упрощающее предположение о том, что задачу можно рассматривать как двумерную. Продольная теплопроводность не рассматривается, т.к. показано, что её влиянием в большинстве практически важных случаев можно пренебречь.